ICARUS AND THE VARIATION OF PARAMETERS

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1. The minor planet 1566, Icarus, also known as 1949 MA and as Baade's object, possesses special interest because of (a) the solar parallax and related quantities, (b) the mass of Mercury, (c) the relativity effect. Of all minor planets known it passes closest to the sun and Mercury. Of all minor planets still "in captivity" it passes closest to the earth. And because of its high eccentricity it presents a severe test both for the relativity motion of its perihelion and for certain aspects of perturbation determination, e.g., for the method of variation of parameters that is to be illustrated herein.

(a) For the solar parallax and the mass of the earth-moon system, whether determined trigonometrically or by the more likely gravitational method, Icarus in 1968 will pass the earth almost at its minimum distance, about four million miles. Thus, roughly, its geocentric parallax will be four times that of Eros in 1931 and the gravitational effects will be sixteen times as great. Balanced against these favorable factors we find Icarus generally too faint to be observed except for short intervals, and then, unless it is very close, only by the largest telescopes. It is often badly placed for observation. In 1951 it was too low in the sky after the end of astronomical twilight for observations to be obtained in either the northern or southern hemisphere. In 1952 it was too low in the northern hemisphere, but happily five observations were obtained at Pretoria, so that the orbit of Icarus is now well established. In the moderately close approaches of 1953, 1958, and 1959 Icarus will again be better placed for southern observatories than for northern. In 1968 it will be favorably placed in both hemispheres and probably brighter than the thirteenth magnitude; unfavorable will be its short visit of only 12 days brighter than 15th magnitude, and rapid motion up to 50″ per minute. Spencer Jones suggests that the latter disadvantage might be overcome in part by a variant of the method of interrupted exposures.[1] He recommends also that the color of Icarus be determined at an early date.

(b) For the mass of Mercury, we find Icarus closer than Venus only about one-eighth of the time, but it can pass Mercury within 8 million miles. By chance the next such passage will be about 1968 May 1, just before the close approach to the earth. It passes within 12 million miles probably every other time; within 20 million miles, probably two times out of three. Moreover, Icarus is sometimes between Mercury and the sun, so that the mass of the former is not in part confused with the latter as it is in Mercury's perturbations on Venus. The ratio of the periods of Icarus and Mercury is 4.645, so that every third return of Icarus will find Mercury in somewhat the same region, with the closest approaches at intervals of 14 or 17 returns, the longer interval being also the 19-year commensurability with the earth. Because the coordinates of Mercury were not conveniently available we elected to postpone the treatment of its perturbations on Icarus.

(c) For the relativity effect on the motion of the perihelion of Icarus we have the figure 10″.05, to be compared with Mercury's well known 43″.03. A recent paper by Gilvarry,[2] however, shows that when the Icarus figure is multiplied by certain factors that attempt to evaluate the observability of the phenomenon it becomes about 14″, as contrasted with Clemence's estimate[3] for Mercury, 3″.03. These estimates are necessarily subjective but suggest that accurate observations of Icarus are justifiable.

2. The ephemerides of Icarus illustrate both the special character discussed in Section 1 and the difficulties that its position and speed often

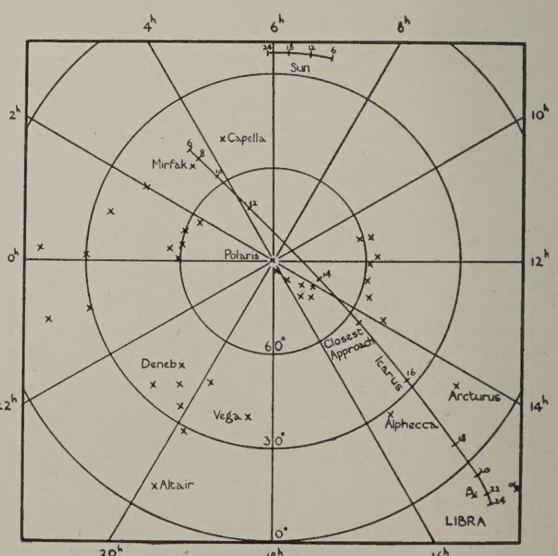

Figure 1. Positions of Icarus and the sun, 1968 June 6 to 24.

TABLE I. OBSERVABILITY EPHEMERIDES

Year	Date	m	ρ	ψ	(+34)	(−26)	$\alpha' \cos \delta$	δ'
1949	June 29	16.0	0.254	160°E	+28°	+89°		
	July 7	16.9	0.364	150 E	+27	+87	− 34′	− 7′
	July 15	17.6	0.480	135 E	+26	+86		
1950	July 28	18.9	0.710	80 E	+30	+60		
	Aug. 7	19.2	0.873	79 E	+24	+62	+ 69	− 46
	Aug. 17	19.6	1.054	78 E	+19	+66		
1951	May 24	20.0	1.501	53 W	—	+33		
	June 13	19.3	1.213	48 W	—	+29	+100	+ 39
	July 3	18.4	1.084	31 W	—	+12		
	Aug. 12	18.6	1.037	37 E	+ 6	+18		
	Sept. 1	19.4	1.233	51 E	+12	+31	+ 91	− 50
	Sept. 21	20.1	1.564	53 E	+10	+34		
1952	June 17	19.9	1.195	78 W	+ 6	+58		
	July 7	19.2	0.875	77 W	+ 7	+58	+113	+ 25
	July 27	18.7	0.667	61 W	+ 6	+40		
1953	Aug. 11	17.8	0.484	114 W	0	+58		
	Aug. 21	17.6	0.411	102 W	—	+46	+129	− 10
	Aug. 31	17.7	0.381	83 W	—	+30		
	Sept. 10	18.0	0.398	60 W	—	+20		
1958	May 17	18.0	0.430	70 W	+34	+50		
	June 6	17.0	0.355	119 W	+37	+82	−100	− 34
	June 26	16.6	0.428	168 W	+22	+83		
	July 16	18.2	0.665	146 E	+21	+80		
1959	June 11	18.8	0.459	25 E	+ 2	—		
	July 1	17.5	0.304	74 E	+45	+50	+141	− 106
	July 21	18.3	0.582	98 E	+37	+80		
1968	June 8	17.0	0.123	32 W	+25	—		
	June 12	15.2	0.063	50 W	+28	—		
	June 16	12.9	0.048*	111 E	+88	+32	+240	−1200
	June 20	13.9	0.100	135 E	+52	+68		
	June 24	14.9	0.165	137 E	+41	+79		

* 1968 minimum distance = 0.042; minimum possible distance = 0.038.

give to the observer. Table I shows the magnitude, m, the geocentric distance in astronomical units, ρ, the angle of elongation, ψ, the maximum altitude when the sun is 18° or more below the horizon for latitudes 34° N. and 26° S., in the columns headed (+34) and (−26), respectively, and approximate values of the apparent velocities $\alpha' \cos \delta$ and δ' in minutes of arc per day. The figures for 1949 through 1952 are shown for comparison with the later ones. Table II shows the apparent positions of Icarus for 1953; and Figure 1, those for 1968.

TABLE II. 1953 FINDING EPHEMERIS

1953	α (1950.0)		δ (1950.0)	
Aug. 7.0	$1^h 41^m.7$	$+10^m.0$	−52°17′	−141′
9.0	1 51.7	+11.8	−54 38	−148
11.0	2 3.5	+13.9	−57 6	−153
13.0	2 17.4	+17.0	−59 39	−158
15.0	2 34.4	+21.2	−62 17	−158
17.0	2 55.6	+26.6	−64 55	−154
19.0	3 22.2	+34.1	−67 29	−140
21.0	3 56.3	+43.3	−69 49	−117
23.0	4 39.6	+52.9	−71 46	− 75
25.0	5 32.5	+59.4	−73 1	− 20
27.0	6 31.9	+59.2	−73 21	+ 44
29.0	7 31.1	+52.4	−72 37	+104
31.0	8 23.5	+42.5	−70 35	+150
Sept. 2.0	9 6.0	+33.3	−68 23	+181
4.0	9 39.3	+25.8	−65 22	+202
6.0	10 5.1	+20.2	−62 0	+220
8.0	10 25.3	+16.0	−58 20	+212
10.0	10 41.3		−54 48	

3. The observations of Icarus thus far at hand and the residuals they show are listed in Table III. The orbit was based upon the observations of 1949 and 1950, but the representation of 1952

TABLE III. OBSERVATIONS AND RESIDUALS

1949 June	α (1950.0)	δ (1950.0)	$\Delta\alpha$ (O−C)	$\Delta\delta$
27.23889	$16^h 41^m 08^s.13$	−27°51′53″.7	$-0^s.08$	0″.0
29.23056	16 31 54.65	−28 16 20.5	+0.01	+0.6
1949 July				
1.31562	16 24 15.33	−28 35 05.3	−0.08	−0.5
13.20660	16 03 10.48	−29 26 09.1	+0.03	+0.1
14.21701	16 02 26.90	−29 28 39.0	+0.06	−0.3
23.19583	16 00 01.70	−29 47 23.4	+0.03	−0.1
24.18750	16 00 06.74	−29 49 16.5	+0.01	+0.1
28.20633	16 00 59.85	−29 56 51.0	+0.02	−0.1
28.22369	16 01 00.01	−29 56 52.4	−0.01	+0.3

TABLE III. OBSERVATIONS AND RESIDUALS—*Continued*

	α (1950.0)	δ (1950.0)	Δα(O – C)	Δδ
1950 Aug.				
3.19167	13 54 21.03	−12 01 25.9	0.00	−1.0
3.20972	13 54 25.84	−12 02 06.5	+0.06	+0.8
4.19167	13 58 42.56	−12 39 36.0	+0.06	−1.3
5.19132	14 02 56.28	−13 16 05.0	−0.01	+0.3
5.20903	14 03 00.65	−13 16 42.1	−0.04	+1.0
1952 June				
23.11944	1 27 12.1	−16 20 47	−0.1	+1
23.14514	1 27 17.6	−16 20 42	−0.4	−2
24.13542	1 31 01.7	−16 15 11	−0.6	−3
1952 July				
25.13750	4 29 06.1	− 9 05 03	−0.2	−6
25.15243	4 29 13.2	− 9 04 35	−0.1	−4

Sources:

1949 June 27 to July 24, Palomar; Nicholson, Richardson, Rule, Baade, *H. A. C.* 1008, 1012, 1017.

1949 July 28, Mt. Wilson; Baade, Nicholson, Richardson, *H. A. C.* 1017.

1950 Aug. 3 to 5, Palomar; Richardson, Nicholson, Harrington, *H. A. C.* 1096.

1952 June and July, Pretoria; Feast, Lourens, Thackeray, *M. N. Astr. Soc. South Africa* 10, 107, 1952.

is so satisfactory as to make differential correction unnecessary at least until after 1953 observations are available.

4. The elements or parameters of the orbit of Icarus are as follows:

$t_0 = 1950$ August $7.0 = 2433500.5$ UT

$M = M_0 + n_0(t - t_0) + \Delta M \qquad M_0 = +0.933740\,30$

$n = n_0 + \Delta n \qquad n_0 = +0.015375\,5380$

$a_x = eP_x = a_{x0} + \Delta a_x \qquad a_{x0} = -0.362753\,59$

TABLE IV. PERTURBATIONS IN THE ELEMENTS, UNIT 10^{-6}

Date	$\frac{t-t_0}{100}$	ΔM	Δa_x	Δa_y	Δa_z	Δb_x	Δb_y	Δb_z	Δa	Δe
1949										
July 3	−4	+1093	+86	+128	+1	+49	+23	+167	+17	+56
Oct. 11	−3	+790	+48	+83	−20	+23	−3	+120	+75	+29
1950										
Jan. 19	−2	+282	+17	+45	−17	+3	−2	+50	+69	+16
Apr. 29	−1	−10	+6	+9	0	−1	+4	+5	+9	+4
Aug. 7	0	0	0	0	0	0	0	0	0	0
Nov. 15	+1	+17	−5	−31	−22	−24	−14	−21	+14	−30
1951										
Feb. 23	+2	−2	+24	−62	−49	−73	−18	−46	+43	−81
June 3	+3	−132	+71	−77	−55	−116	−9	−62	+59	−117
Sept. 11	+4	−229	+89	−74	−54	−118	0	−59	+30	−122
Dec. 20	+5	−203	+105	−55	−48	−111	+15	−48	+3	−111
1952										
Mar. 29	+6	−102	+117	−47	−50	−116	+21	−44	+4	−112
July 7	+7	−36	+131	−35	−50	−124	+27	−38	+21	−109
Oct. 15	+8	−84	+155	−48	−21	−121	+25	−23	+18	−114
1953										
Jan. 23	+9	−94	+167	−19	−6	−107	+43	−10	−5	−91
May 3	+10	−102	+156	−4	+6	−90	+52	−9	−16	−68
Aug. 11	+11	−7	+140	+9	+18	−75	+56	−10	−13	−45
Nov. 19	+12	+25	+137	+7	+31	−76	+57	−17	+4	−39
1954										
Feb. 27	+13	−53	+133	+14	+37	−72	+59	−17	+15	−28
June 7	+14	−183	+116	+21	+48	−60	+63	−25	+9	−10
Sept. 15	+15	−209	+103	+22	+58	−52	+67	−37	+1	+2
Dec. 24	+16	−221	+99	+24	+66	−54	+67	−40	+30	+9
1955										
Apr. 3	+17	−400	+77	+5	+60	−61	+49	−54	+72	+1
July 12	+18	−626	+69	−2	+59	−65	+48	−66	+76	0
Oct. 20	+19	−853	+65	−12	+63	−68	+49	−81	+65	−4
1956										
Jan. 28	+20	−986	+55	−21	+70	−62	+48	−90	+48	−2
May 7	+21	−1140	+39	−41	+62	−70	+34	−102	+72	−14
Aug. 15	+22	−1284	+44	−57	+49	−88	+28	−114	+83	−35
Nov. 23	+23	−1503	+48	−66	+48	−97	+29	−125	+80	−43
1957										
Mar. 3	+24	−1683	+55	−72	+50	−101	+32	−129	+68	−50
June 11	+25	−1763	+64	−70	+46	−102	+37	−127	+44	−54
Sept. 19	+26	−1791	+75	−83	+33	−119	+33	−134	+50	−75
Dec. 28	+27	−1863	+88	−84	+29	−130	+36	−135	+58	−83
1958										
Apr. 7	+28	−1998	+98	−88	+29	−137	+39	−137	+49	−91
July 16	+29	−1981	+131	−62	+37	−130	+63	−120	−5	−83
Oct. 24	+30	−1870	+141	−45	+41	−124	+76	−113	−23	−72
1959										
Feb. 1	+31	−1734	+149	−34	+42	−125	+81	−109	−15	−67
May 12	+32	−1692	+153	−30	+44	−128	+84	−108	−6	−65
Aug. 20	+33	−1672	+156	−25	+51	−123	+89	−107	−18	−59

$$
\begin{aligned}
a_y &= eP_y = a_{y0} + \Delta a_y & a_{y0} &= +0.598282\,70\\
a_z &= eP_z = a_{z0} + \Delta a_z & a_{z0} &= +0.440160\,94\\
b_x &= e\sqrt{p}Q_x = b_{x0} + \Delta b_x & b_{x0} &= -0.390922\,85\\
b_y &= e\sqrt{p}Q_y = b_{y0} + \Delta b_y & b_{y0} &= -0.278029\,54\\
b_z &= e\sqrt{p}Q_z = b_{z0} + \Delta b_z & b_{z0} &= +0.055733\,24
\end{aligned}
$$

The differences Δa_x, etc., with the exception of Δn, are given in Table IV at intervals of 100 days and to two fewer places than the foregoing constants, together with derived values of Δa and Δe defined as follows:

$$
\begin{aligned}
a &= a_0 + \Delta a & a_0 &= +1.077707\\
e &= e_0 + \Delta e & e_0 &= +0.826604
\end{aligned}
$$

From the tabulated quantities we may obtain also

$$
\begin{aligned}
\Delta\Omega &= -59^\circ\!.4\,\Delta a_x + 57^\circ\!.9\,\Delta a_y - 127^\circ\!.7\,\Delta a_z\\
&\qquad + 60^\circ\!.9\,\Delta b_x - 59^\circ\!.4\,\Delta b_y + 130^\circ\!.9\,\Delta b_z\\
\Delta i &= +13^\circ\!.9\,\Delta a_x - 13^\circ\!.5\,\Delta a_y + 29^\circ\!.9\,\Delta a_z\\
&\qquad + 39^\circ\!.7\,\Delta b_x - 38^\circ\!.7\,\Delta b_y + 85^\circ\!.4\,\Delta b_z\\
\Delta\omega &= -\Delta v - 0.921\,\Delta\Omega
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta v &= +143^\circ\!.3(a_{x0}\Delta b_x + a_{y0}\Delta b_y + a_{z0}\Delta b_z)\\
&= -143.3(b_{x0}\Delta a_x + b_{y0}\Delta a_y + b_{z0}\Delta a_z)
\end{aligned}
$$

for use with

$$
\begin{aligned}
\Omega &= \Omega_0 + \Delta\Omega & \Omega_0 &= 87^\circ\!.7464\\
i &= i_0 + \Delta i & i_0 &= 22^\circ\!.9788\\
\omega &= \omega_0 + \Delta\omega & \omega_0 &= 30^\circ\!.9115
\end{aligned}
$$

The integration tables from which the quantities in Table IV were determined maintain a 10-day interval throughout most of the calculation, with reduction at perihelion passage to 2 days in 1950 and 1951 and to an even shorter interval, because of a close approach to Venus, in 1952. Calculations at the reduced interval have not yet been undertaken for subsequent perihelion passages, but must be before observation residuals after 1953 can be regarded as entirely consistent with the basic orbit.

5. Minimum formulae and a sample calculation, for problems involving moderate or large eccentricities, such as that of Icarus, are as follows:

Given values of M, possibly n, and at least five of $a_x = eP_x$, $b_x = e\sqrt{p}Q_x$, a_y, b_y, a_z, b_z, which have been obtained by numerical integration from tables, or which are two-body values at the start of a step-by-step integration or of successive approximations by quadratures. For Icarus, on

$$
\begin{aligned}
t &= \text{1952 January 9.0} = 2434020.5\ \text{U.T.}\\
a_x &= -0.36265 & b_x &= -0.39103\\
a_y &= +0.59823 & b_y &= -0.27801\\
a_z &= +0.44011 & b_z &= +0.05569\\
n &= +0.0153755 & M &= +2.64564\ \text{(radians)}
\end{aligned}
$$

Note that b_y may be thought of as having been obtained not by integration but from

$$a_x b_x + a_y b_y + a_z b_z = 0 \qquad (1)$$

This equation should be used to obtain preferably the b that has the largest a as coefficient. I have preferred to integrate all of the b's, however, using equation (1) for check. This check is an exceptionally good one for finding errors in the tables and should be used every ten steps or so after enough central differences are available to insure full accuracy in the integrals. With n also integrated, however, a good check will be found in equation (6).

$$
\begin{aligned}
a_x^2 + a_y^2 + a_z^2 = e^2 &= 0.683091 && (2)\\
e &= 0.82649\\
(b_x^2 + b_y^2 + b_z^2)/e^2 = p &= 0.341529 && (3)\\
p/(1 - e^2) = a &= 1.07769 && (4)\\
a &= 1.03812\\
a^{\frac{3}{2}} &= 1.11877
\end{aligned}
$$

Then, with $k = 0.01720209895$ for the Gaussian constant, the mass of Icarus $= 0$, and the mass of Mercury, $m_☿ = 0.000000\,163$, included with the sun's mass,

$$
\begin{aligned}
k\sqrt{1 + m_☿} = k_m &= 0.017202\,10036 && (5)\\
k_m/a^{\frac{3}{2}} = n &= 0.0153759 && (6)
\end{aligned}
$$

Equation (6) is here used for check, but could be a primary source of n. Then

$$M = E - e\sin E = +2.64564 \qquad (7)$$

is solved for E by whatever technique the computer prefers. In this example

$$
\begin{aligned}
E &= +2.86853 & \cos E &= -0.96295\\
\sin E &= +0.26968 & e\cos E &= -0.79587
\end{aligned}
$$

$$
\begin{aligned}
a(1 - e\cos E) = r &= +1.93539 && (8)\\
\sqrt{a}\,e\sin E = r\dot{r} = u &= +0.23138 && (9)\\
r - p = w &= +1.59385 && (10)\\
\frac{e\cos E}{r} = \frac{1}{r} - \frac{1}{a} = \dot{u} &= -0.41122 && (11)\\
u/r = \dot{r} = \dot{w} &= +0.11955 && (12)\\
u\dot{w} - w\dot{u} = e^2 &= 0.683084^* && (13)\\
(ub_x - wa_x)/e^2 = x &= +0.71372 && (14)\\
y &= -1.49003\\
z &= -1.00806
\end{aligned}
$$

$$(\dot{u}b_x - \dot{w}a_x)/e^2 = \dot{x} = +0.29887 \qquad (15)$$
$$\dot{y} = +0.06266$$
$$\dot{z} = -0.11055$$
$$x^2 + y^2 + z^2 = r^2 = \quad 3.74577^* \qquad (16)$$
$$x\dot{x} + y\dot{y} + z\dot{z} = r\dot{r} = u = +0.23139^* \qquad (17)$$
$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{s}^2 \qquad * \qquad (18)$$
$$\dot{s}^2 - 1/r = \dot{u} \qquad * \qquad (19)$$
$$up = xb_x + yb_y + zb_z \qquad * \qquad (20)$$
$$-w = xa_x + ya_y + za_z \qquad * \qquad (21)$$
$$a_x = \dot{u}x - u\dot{x} \qquad * \qquad (22)$$
$$b_x = \dot{w}x - w\dot{x} \qquad * \qquad (23)$$

An asterisk indicates a check equation. Where no numerical value is given the check was not used in the actual calculation.

The following masses were adopted for the perturbation terms, Mercury being included only as augmenting the solar mass:

Sun + Mercury + Icarus:	M = 1.000000 163
Venus:	.000002 45
Earth:	.000003 03577
Mars:	.000000 32
Jupiter:	.000954 79
Saturn:	.000285 58
Uranus:	.000043 73
Neptune:	.000051 78

The calculation for Jupiter alone based on heliocentric coordinates of Jupiter from *Planetary Coordinates, 1940–1960*, p. 68, will be indicated as follows:

$$x_J = +4.70316$$
$$y_J = +1.45432$$
$$z_J = +0.50892$$
$$1/r_J^3 = \quad 0.008249\,28$$
$$m_J/Mr_J^3 = \quad 0.000007\,8763$$
$$x_J - x = x_{IJ} = +3.98944$$
$$y_{IJ} = +2.94435$$
$$z_{IJ} = +1.51698$$
$$x_{IJ}^2 + y_{IJ}^2 + z_{IJ}^2 = r_{IJ}^2 = \quad 26.88606$$
$$m_J/Mr_{IJ}^3 = \quad 0.000006\,8488$$

In the remainder of this tabulation numerical values will be expressed in units of 10^{-8}.

$$\frac{m_J}{M}\left\{\frac{x_{IJ}}{r_{IJ}^3} - \frac{x_J}{r_J^3}\right\} = [\ddot{x}]_J = -\ 972.05$$
$$[\ddot{y}]_J = +\ 871.07$$
$$[\ddot{z}]_J = +\ 638.12$$
$$[\ddot{x}]_V + [\ddot{x}]_E + [\ddot{x}]_M + [\ddot{x}]_J + \cdots$$
$$= [\ddot{x}] = -\ 408$$
$$[\ddot{y}] = +\ 754$$
$$[\ddot{z}] = +\ 579$$
$$x[\ddot{x}] + y[\ddot{y}] + z[\ddot{z}] = [\dot{u}] = -1998$$
$$2(\dot{x}[\ddot{x}] + \dot{y}[\ddot{y}] + \dot{z}[\ddot{z}]) = [\ddot{u}] = -\ 277$$
$$2u[\dot{u}] - r^2[\ddot{u}] = [\dot{w}] = +\ 113$$
$$[\dot{u}]/r = [\ddot{w}] = -1033$$
$$[\ddot{u}]x - [\dot{u}]\dot{x} - u[\ddot{x}] = [\dot{a}_x] = +\ 494$$
$$[\dot{a}_y] = +\ 363$$
$$[\dot{a}_z] = -\ \ 76$$
$$[\ddot{w}]x - [\dot{w}]\dot{x} - w[\ddot{x}] = [\dot{b}_x] = -\ 121$$
$$[\dot{b}_y] = +\ 330$$
$$[\dot{b}_z] = +\ 131$$

TABLE V. SAMPLE OF INTEGRATION TABLES, UNIT 10^{-8}

Date 1951–52	$\Sigma^2[\dot{n}]$	$\Sigma[\dot{n}]$	$[\dot{n}]$	$\delta[\dot{n}]$	$\delta^2[\dot{n}]$	$\delta^3[\dot{n}]$
Dec. 10	−8498.05					
		−39.19		−2.94		−.12
20	−8537.24		+12.22		+.24	
		−26.97		−2.70		−.17
30	−8564.21		+ 9.52		+.07	
		−17.45		−2.63		
Jan. 9	−8581.66		+ 6.89			
		−10.56				
19	−8592.22					

Date 1951–52	$\Sigma[\dot{f}]$	$[\dot{f}]$	$\delta[\dot{f}]$	$\delta^2[\dot{f}]$	$\delta^3[\dot{f}]$	$\Sigma[\dot{f}]$	$[\dot{f}]$	$\delta[\dot{f}]$	$\delta^2[\dot{f}]$	$\delta^3[\dot{f}]$
	$[\dot{f}] = [\dot{M}]$					$[\dot{f}] = [\dot{a}_z]$				
	−34128		+320		−16	−27903		− 90		+ 6
Dec. 20		+3271		+128			+ 78		+ 7	
	−30857		+448		−26	−27825		− 83		+ 5
30		+3719		+102			− 5		+12	
	−27138		+550			−27830		− 71		
Jan. 9		+4269					− 76			
	−22869					−27906				
	$[\dot{f}] = [\dot{a}_x]$					$[\dot{f}] = [\dot{b}_x]$				
	+60756		− 1		− 8	−64685		−117		+ 8
Dec. 20		+ 386		+ 39			+108		− 2	
	+61142		+ 38		− 7	−64577		−119		+11
30		+ 424		+ 32			− 11		+ 9	
	+61566		+ 70			−64588		−110		
Jan. 9		+ 494					−121			
	+62060					−64709				
	$[\dot{f}] = [\dot{a}_y]$					$[\dot{f}] = [\dot{b}_z]$				
	−32133		−133		+10	−27856		− 72		+ 4
Dec. 20		+ 534		+ 31			+198		+25	
	−31599		−102		+ 2	−27658		− 47		+ 2
30		+ 432		+ 33			+151		+27	
	−31167		− 69			−27507		− 20		
Jan. 9		+ 363					+131			
	−30804					−27376				

$$(a_x[\dot{b}_x] + a_y[\dot{b}_y] + a_z[\dot{b}_z])/e^2 = [\dot{v}]\sqrt{p} = +\ 438$$

$$-(b_x[\dot{a}_x] + b_y[\dot{a}_y] + b_z[\dot{a}_z])/e^2 = [\dot{v}]\sqrt{p} = +\ 437^*$$

$$([\dot{v}]\sqrt{p} - 2[\dot{u}])/\sqrt{a} = [\dot{M}] = +4270$$

$$-\tfrac{3}{2}na[\ddot{u}] = [\dot{n}] = +\quad 6.89$$

$$-(x[\dot{a}_x] + y[\dot{a}_y] + z[\dot{a}_z]) = [\dot{w}] = +\ 112^*$$

$$(x[\dot{b}_x] + y[\dot{b}_y] + z[\dot{b}_z] + u[\dot{w}])/p = [\dot{u}] \qquad *$$

$$2(\dot{x}[\dot{a}_x] + \dot{y}[\dot{a}_y] + \dot{z}[\dot{a}_z] + \dot{s}^2[\dot{u}])/u = [\ddot{u}] \qquad *$$

$$-\{w[\dot{u}] + \tfrac{1}{2}au[\ddot{u}](p + r)\}/a^{\frac{3}{2}}e^2 = [\dot{M}] = +4269^*$$

The variations $[\dot{n}]$, $[\dot{M}]$, and at least five of $[\dot{a}_x]$, $[\dot{a}_y]$, $[\dot{a}_z]$, $[\dot{b}_x]$, $[\dot{b}_y]$, $[\dot{b}_z]$ are now entered into the tables, and the sums and backwards differences determined, so that the ends of the tables appear as in Table V.

The table for $[\dot{b}_y]$ is not included among the foregoing because it is not strictly necessary in a minimum calculation, but it has always been carried as a check, and I strongly recommend that it should be so carried.

Correction to M and the starting elements are obtained from these tables by means of the following formulae:

$$\Delta M_i = k_m \int\int_{t_0}^{t_i} [\dot{n}]dt^2 + k_m \int_{t_0}^{t_i} [\dot{M}]dt \quad (24)$$

$$\Delta f_i = k_m \int_{t_0}^{t_i} [\dot{f}]dt, \quad f = n, a_x, b_x, \text{etc.} \quad (25)$$

where, for step-by-step integration, when central differences are not available,

$$k_m \int\int_{t_0}^{t_i} [\dot{n}]dt^2 = w^2k_m\left\{\Sigma^2[\dot{n}]_i + \frac{1}{12}[\dot{n}]_{i-1} + \frac{1}{12}\delta[\dot{n}]_{i-\frac{3}{2}} + \frac{19}{240}\delta^2[\dot{n}]_{i-2} + \frac{18}{240}\delta^3[\dot{n}]_{i-\frac{5}{2}} + \cdots\right\} \quad (26)$$

$$k_m \int_{t_0}^{t_i} [\dot{f}]dt = wk_m\left\{\Sigma[\dot{f}]_{i-\frac{1}{2}} + \tfrac{1}{2}[\dot{f}]_{i-1} + \frac{5}{12}\delta[\dot{f}]_{i-\frac{3}{2}} + \frac{3}{8}\delta^2[\dot{f}]_{i-2} + \frac{251}{720}\delta^3[\dot{f}]_{i-\frac{5}{2}} + \cdots\right\},$$
$$f = M, n, a_x, b_x, \text{etc.} \quad (27)$$

In the present example only one, two, or three terms of these expressions are significant. With the initial elements for $t_0 = 2433500.5$ UT, as shown in Section 4, and

$$M = M_0 + n_0(t - t_0) + \Delta M \quad (28)$$

$$f = f_0 + \Delta f, \quad f = n, a_x, b_x, \text{etc.} \quad (29)$$

the example yields, on 1952 January 19.0 = 2434030.5 UT,

ΔM	$= -0.000183$	M	$= \ \ 2.79941$
Δn	$= -0.00000001$	n	$= \ \ 0.0153755$
Δa_x	$= +0.000107$	a_x	$= -0.36265$
Δa_y	$= -0.000053$	a_y	$= +0.59823$
Δa_z	$= -0.000048$	a_z	$= +0.44011$
Δb_x	$= -0.000111$	b_x	$= +0.39103$
		b_y	$= -0.27801$
Δb_z	$= -0.000047$	b_z	$= +0.05569$

and the calculations are ready for the next step. It will be noted that the foregoing derived values, except M, are identical with those used at the start of the step illustrated. It was necessary in fact to use the integration formulae only for every fourth or fifth step of the calculation. The interval of the tables was determined by the run of the differences and not by interpolation or integration.

6. Comparisons between the method of variation of parameters and the methods of Cowell and Encke, and between the formulae of Section 5 and older formulae for the variation of parameters will illuminate special perturbation theory and its practical application to objects such as Icarus.

Cowell's method, as is well known, involves the integration of derivatives that are larger and that have larger differences than those of Encke's method or the method of variation of parameters. The contrast becomes more pronounced, if the orbit is eccentric, as the object approaches perihelion. In these circumstances Cowell's method is customarily replaced by one of the other two, but it is profitable, nevertheless, to consider some of the differences that would appear if we attempted to bring an integration of Icarus to perihelion without diminishing a 10-day interval. Table VI supplies the second derivative of x and its differences as they would appear in such an attempt by Cowell's method, together with the derivative of one of the parameters and the differences of this derivative as they appear in the method of variation of parameters.

It will be noted at a glance that the sizes of the derivatives and differences favor the latter

method. Draper found it impossible to carry Icarus by perihelion with an interval one-fourth as great, in the work upon which the first calculations of the present paper were based.[7]

My purpose in reproducing Table VI, however, is to show not so much the contrast between the sizes of the integrands and their differences, as the similarity of the oscillation patterns of the two sets of differences. The rapid change in the position of Icarus at perihelion, 180° in about 10 days, is clearly reflected in the perturbations as well as in the principal or solar terms of the accelerations. The differences of the variations of the parameters, though small, are yet erratic enough in their localized oscillation at the selected 10-day interval to cause uncertainty in subsequent values of the integrated parameters, and we are forced to a 2-day interval to get by perihelion. It is evident, however, that the uncertainty is too small to affect the calculated values of the variations for some distance beyond perihelion. We are thus able to carry well past with the 10-day interval, compute all at once the values of the parameters at the shorter interval, then integrate by quadratures to obtain the necessary correction to the second sum or integral in the 10-day tables.

TABLE VI. COWELL VS. VARIATION OF PARAMETERS

	$\frac{1}{k m^2}\frac{d^2x}{dt^2}$	δ	δ^2	δ^3	δ^4
June 13	− 1.0053867		− 0.2791412		− 0.3874407
		− 0.5697825		− 0.5461929	
23	− 1.5751692		− 0.8253341		− 1.6677127
		− 1.3951166		− 2.2139056	
July 3	− 2.9702858		− 3.0392397		+ 36.9632855
		− 4.4343563		+34.7493799	
13	− 7.4046421		+31.7101402		−109.8118522
		+27.2757839		−75.0624723	
23	+19.8711418		−43.3523321		+131.7093642

	$\frac{1}{k m}\frac{da_y}{dt}$	δ	δ^2	δ^3	δ^4
June 13	−0.00000230		− 69		+ 35
		+ 9		+ 14	
23	−0.00000221		− 55		+ 136
		− 46		+150	
July 3	−0.00000267		+ 95		+ 6
		+ 49		+156	
13	−0.00000218		+251		− 736
		+300		−580	
23	+0.00000082		−329		+1151

The amount of labor necessary to find corrections of about 2 units in the seventh decimal place, due to the rapid motion of the perturbed body rather than to its proximity to disturbing bodies, nevertheless indicates that it is urgent to find other solutions to the difficulty. Three avenues deserving of exploration are: (1) In the mathematics of finite differences to investigate means to correct for the effect of the oscillating differences of a near-singularity or the extent to which they can be tolerated. (2) On the astronomical side to attempt to extend the variation of parameters to include a part of the perturbations as well as the solar terms of the equations of motion. (3) The substitution of another independent variable for t, to which possibility I shall devote some discussion in the next section.

A 20-day or 40-day interval would have been possible in the neighborhood of aphelion, but the 10-day interval was continued because of the shortness of the period of Icarus (409 days).

Two further comparisons with Cowell's method: (1) The computers found the labor of computing the variations of the parameters from the perturbations small in comparison with that of computing the perturbations themselves from the coordinates of the seven disturbing planets. (2) First-order perturbations cannot be obtained by one double-summation "quadrature" in Cowell's method; at least the principal or solar terms must be determined step-by-step or by successive approximations. In the variations of the parameters, on the other hand, these terms are eliminated, so that first-order "quadrature" integrations without successive approximations are possible.

The foregoing two comments apply also to a comparison with Encke's method, whose periodic rectifications and gradually increasing $(fqx - \xi)$ terms should be balanced against the calculation of the variations of the parameters. Otherwise, in step-by-step integration, the problems of differencing and intervals will be much alike in the two methods. In first-order integrations, however, Encke's method is in the same relatively disadvantageous position as Cowell's method.

The first and most obvious comparison between the formulae of Section 5 and the older formulae, as shown by Watson,[5] Oppolzer,[6] Bauschinger,[7] and others, is that the former refer to trigonometrical tables only for functions of the eccentric anomaly while the latter refer also for v, ω or u, i, and, for each disturbing planet, b and $l - \Omega$. The integration of a_x, b_x, etc., moreover, eliminates much of the calculation involved in the

combinations of sines and cosines into the components of orientation vectors. Perhaps even more important are the checks supplied by the integration of n and the third b; no such checks have been used or are readily available with the older formulae.

In problems involving a resisting medium, of which there may be more in the future, the new formulae are by chance at an advantage in that they involve the calculation of the components of the velocity, $\dot{x}$, $\dot{y}$, $\dot{z}$, rather than the components of the transverse vector (in the orbit plane at right angles to the radius vector).

The way in which each set of formulae meets the special requirements of very small eccentricities is a matter of some interest, though better reserved to another paper than discussed at length here. I shall remark only that both formulations replace the mean anomaly, M, by the mean longitude, L, because $[\dot{M}]$ has an e in the denominator, effectively first power only, and $[\dot{L}]$ does not. In the formulae of Section 5, moreover, we must replace the equations for $[\dot{b}_x]$, $[\dot{b}_y]$, $[\dot{b}_z]$ by

$$\begin{aligned} [\dot{c}_x] &= y[\ddot{z}] - z[\ddot{y}] \\ [\dot{c}_y] &= z[\ddot{x}] - x[\ddot{z}] \\ [\dot{c}_z] &= x[\ddot{y}] - y[\ddot{x}] \end{aligned} \qquad (30)$$

where $c_x = \sqrt{p}R_x$, etc., are free from the e in $b_x = e\sqrt{p}Q_x$, etc., so that the components of the orientation vectors may be more accurately determined. The uncertainty remaining in $a_x = eP_x$, etc., has its counterpart in the older formulation in the determination of π; the position of perihelion is thus made uncertain, but not the position of the object under investigation. In this formulation I have been considerably anticipated by Milankovitch.[8]

7. Modifications or additions in or to the minimum formulae of Section 5 may be sought to avoid certain disadvantages, but will in turn introduce certain others. Before discussing possible changes I list both the disadvantages present in the minimum formulae and those introduced by changes:

a) No check upon the integration of M. It will be noted that equations (1) and (6) provide checks for a_x, a_y, a_z, b_x, b_y, b_z, and n, but that there is no final check, other than recalculation, on the integration of $[\dot{M}]$ or even of $[\dot{n}]$ by (27) and (26), respectively.

b) Solution of Kepler's equation. I should note that the computers to whom I give credit at the end of this paper found Kepler's equation unobjectionable, especially as compared with the devices proposed to avoid it.

c) Changes of interval required by perihelion passage rather than perturbations (cf. Section 6).

d) Required interval even shorter than that of the minimum formulae.

e) Interpolation of planetary coordinates when the arguments for the tables do not coincide with the dates in *Planetary Coordinates*.

f) Said interpolation not possible in advance.

g) Reinterpolation necessary for each approximation.

h) Accurate evaluation of n required before the tables of M or t can be extended.

Table VII indicates how these disadvantages are found in the several devices that have come to my attention. Brief notes are given describing these devices, but a more complete analysis of the evaluation is left to the reader.

TABLE VII. COMPARISON OF ADVANTAGES AND DISADVANTAGES IN VARIOUS INTEGRATION PROCEDURES

		a	b	c	d	e	f	g	h
I.	Minimum formulae, as above.	x	x	x					
II.	Integrate M_0 or T.	x	x	x					
	I + II.		x	x					
III.	Integrate E or other variable.	x		x	x				
	I + III.			x	x				
IV.	M as independent variable.	x		x		x	x	x	x
	II + IV.			x		x	x	x	x
V.	E as independent variable.					x	x	x	x
VI.	ϵ as independent variable.					x			

NOTES ON TABLE VII

II. The integration of M_0 or T, treated as parameters, is based upon the equations,

$$[\dot{M}_0] = [\dot{M}] - [\dot{n}](t - t_0), \qquad (31)$$

$$[\dot{T}] = \{[\dot{n}](t - T) - [\dot{M}]\}/n. \qquad (32)$$

The latter of these equations I included in my first discussion of the method of variation of parameters.[9] The use of either is compared to the usual procedure, as set forth in Section 6 above, in my more recent paper.[10] We made use of one or the other of these procedures to provide a check upon M throughout most of our work on Icarus.

III. The integration of the eccentric or of the true anomaly for Icarus, to avoid the solution of Kepler's equation, cannot be carried on with an interval larger than can be used with Cowell's method. Experiments with $N = E - e \sin N$, and with other such variables whose differences might be expected to be smaller than those of E, were disappointing.

IV. The use of M as independent variable has been discussed recently by Merton.[11] The perturbative variations are to be multiplied by, and t is to be integrated from

$$\frac{dt}{dM} = \frac{1}{n}\left\{1 - k_m[\dot{M}]\frac{dt}{dM}\right\} \qquad (33)$$

as Merton formulated it, or, more simply and avoiding the possible need for a second approximation, by

$$\frac{dt}{dM} = \{n + k_m[\dot{M}]\}^{-1}. \qquad (34)$$

This procedure avoids the solution of Kepler's equation by tabulating certain functions for integral values of M and e,

as proposed originally by Crommelin for first-order quadratures.[12]

V. The use of E as independent variable requires that the perturbative variations be multiplied by and t be integrated from

$$\frac{dt}{dE} = \frac{r}{a}\,\{n + k_m[\dot{M}]+k_m[\dot{e}]\sin E\}^{-1}. \tag{35}$$

The differences are well behaved and the interval of t is automatically shortened around perihelion.

VI. If we designate by ϵ the eccentric anomaly in a fixed reference orbit, let us say the initial osculating one, t is obtained without integration or approximation from

$$t - t_0 = \frac{\epsilon - e_0 \sin \epsilon - M_0}{n_0} \tag{36}$$

and the perturbative accelerations are multiplied by

$$\frac{dt}{d\epsilon} = \frac{1 - e_0 \cos \epsilon}{n_0}. \tag{37}$$

Both M and E would be integrated to check through Kepler's equation. For comparison (orbit of Icarus): If the interval of ϵ is 10°, there are 36 steps per revolution and the interval of t varies from 1.96 days at perihelion to 20.7 days at aphelion. In our calculations by the "minimum formulae" there were 41 ten-day steps per revolution plus about 25 two-day steps to carry the integrals by perihelion, a total of 66 steps. Since the 25 two-day steps also required interpolation, though of a somewhat simpler character than that considered under disadvantage e), it seems that the use of ϵ as independent variable should be seriously considered at least for objects whose orbits are as eccentric as that of Icarus. In first-order quadratures ϵ is synonymous with E and the proposed device, requiring no tables, appears to be considerably simpler than Crommelin's.[12]

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